27 Reasoning in Mathematical Development

Neurocognitive Foundations and Their Implications for the Classroom

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Relations are ubiquitous in mathematics, from the understanding of measurement and patterns to the acquisition of algebra and fractions. In line with this observation, a growing body of literature indicate that individual differences in mathematical skills are associated with individual differences in the ability to reason about relations. In the present chapter, we review these studies and discuss what is known about the neural and behavioral development of two major forms of relational reasoning (i.e., transitive reasoning and analogical reasoning). We argue that relational reasoning may not only relate to mathematical skills because both place demands on common general cognitive resources, but also because relational reasoning and numerical skills share some underlying neurocognitive representations. Finally, the educational implications of these studies are discussed. Notably, we suggest that teachers may help scaffold the development of relational reasoning skills in the classroom by promoting situations in which children are engaged in problem-solving.

In one of his many clever experiments, the pioneering developmental psychologist Jean Piaget asked several five- and six-year-olds to sell him some candies (Piaget, 1952). For each coin a child would get, Piaget would receive one candy. Children did not have much difficulty understanding this one for one exchange. Given that the number of coins never exceeded children's counting ability, participants were also able to tell Piaget how many coins they had gained at the end of the trade. However, when children were asked to determine the number of candies that Piaget had received in exchange for the coins, they struggled to answer. In other words, children could not infer that if there was one coin for one candy, the number of coins and candies had to be identical.

This classic experiment illustrates the role inference making may play in the development of mathematical skills. Mathematical development is more than learning quantitative concepts. It is also learning to manipulate and to make inferences based on these concepts. In other words, mathematical development requires reasoning. It may thus appear surprising that relatively few developmental psychology studies have investigated how reasoning skills contribute to mathematical development. Part of the explanation may lie in the breakthrough discovery that, contrary to Piaget's assumptions, children do possess non-verbal mechanisms providing them with early intuitions about quantitative information (see Chapter 11). This has propelled investigations into young children's numerical knowledge, as well as into the extent to which early non-symbolic intuitions may underlie symbolic math skills (Feigenson et al., 2013). However, the finding that children may have domain-specific quantitative skills does not imply that a domain-general skill such as

This research was supported by a grant from the Agence Nationale de la Recherche (ANR-14-CE30–0002).

reasoning cannot contribute to mathematical development. In fact, there is increasing awareness in the cognitive literature that emerging mathematical skills in children are supported by various domain-general abilities, including working memory, executive control and attention (Fias et al., 2013; Houdé et al., 2011; Houdé, 2019). In keeping with this growing body of research, several studies suggest that reasoning skills also support several aspects of mathematical learning (Inglis & Attridge, 2017; Morsanyi & Szücs, 2014; Richland & Simms, 2015; Singley & Bunge, 2014). This may especially be the case of a particular type of reasoning that will be the focus of this chapter (i.e., reasoning about relations).

As stated in the previous paragraph, relations are ubiquitous in mathematics. For instance, early stages of mathematics education require children to use and combine words expressing relations (e.g., large/small, high/low, long/short) to compare sizes. Later on, understanding relations between numbers and operations (e.g., addition is inversely related to subtraction) is critical to master arithmetic. Relations are also central to algebra. That is, algebraic equations essentially indicate an equal relationship between two expressions in which numbers are related to variables. Finally, the importance of understanding relations between quantities is obvious when children learn fractions, which are defined by the relationship between the numerator and the denominator.

Relational reasoning, or the ability to attend to and manipulate relations, is fundamental to all of the activities described above. Sections 27.1 and 27.2 discuss two of the main types of relational reasoning involved in mathematical learning: transitive reasoning and analogical reasoning. We first describe what is meant by transitive reasoning, its relationship to mathematical learning, and subsequently turn to analogical reasoning. We end the chapter with a discussion of how to promote relational reasoning during mathematics learning.

27.1 Transitive Reasoning

Transitivity is a property that arises from a set of items that can be ordered along a single continuum. A relation is said to be "transitive" when it allows reasoners to infer a relationship between two items (e.g., A and C) from two other overlapping pairs (e.g., A and B; B and C). For example, the relation "older than" is transitive because it allows for the following type of inference:

(1) Ann is older than Tom.
Tom is older than Bill.
Therefore, Ann is older than Bill

This inference is based on an ordering of items along a linear continuum. However, transitive relations are not necessarily linear. For instance, transitive inferences can also be made from sets that can be included in one another, such as in the inference in (2):

(2) All tulips are flowers.
All flowers are plants.
Therefore, all tulips are plants.

As is clear from these examples, a transitive conclusion follows out of necessity from the premises. In other words, if the premises are true, the conclusion is necessarily true. This is the very definition of a "deduction," and this is why transitive reasoning is typically considered an instance of deductive reasoning.

The ability to recognize transitive relations and make associated inferences may contribute to the acquisition of many mathematical concepts. For example, transitive reasoning facilitates the extraction of ordinal information from sets of items and supports the understanding of hierarchical classification (Kallio, 1988; Newstead et al., 1985; Piaget & Inhelder, 1967; Rabinowitz & Howe, 1994). It is also an integral part of measurement skills in children (Bryant & Kopytynska, 1976; Piaget & Inhelder, 1967; Wright, 2001). These observations naturally suggest that transitive reasoning skills might be associated with mathematical performance. Several recent studies tested this hypothesis. For instance, Handley et al. (2004) asked children from nine to eleven years old to evaluate the validity of transitive inferences such as the one in (1). Importantly, the content of the arguments was arbitrary so that children did not have prior beliefs regarding the conclusion.[1] The authors found that reasoning accuracy on these neutral arguments was positively related to performance on a standardized mathematics test and a teacher-administered measure of numeracy. These relationships were specific to mathematical skills because no significant correlation was observed between reasoning accuracy and teacher-administered measures of reading or writing.

An issue with the study above is that transitive arguments were intermixed with conditional arguments (e.g., arguments of the form "If P then Q, there is a P, therefore there is a Q"). Because performance associated with transitive arguments was not separated from performance associated with conditional arguments, it is unclear to what extent the relationships observed in Handley et al. (2004) are specific to transitive inferences. Two subsequent studies address this issue. Specifically, Morsanyi et al. (2017a, 2017b) found a specific relationship between transitive reasoning performance and mathematical abilities in adolescents and adults, respectively. It was also found that adult participants who were the most accurate at assessing the validity of a transitive inference were also the most accurate at positioning a given number on a bounded line with labeled endpoints (the so-called number-line task) (Morsanyi et al., 2017b). This latter finding raises the possibility that transitive reasoning and numerical processing may share some underlying mechanisms.

Some important evidence for the idea that common cognitive mechanisms underlie both transitive reasoning and numerical processing comes from neuroimaging studies. Indeed, processing transitive relations and processing numerical information appear to rely on the same region of the posterior parietal cortex, i.e., the intraparietal sulcus (IPS). On the one hand, the IPS is systematically activated in a wide range of numerical tasks, whether those involve quantity comparison (Ansari, 2008, see Chapter 28), mental arithmetic (Peters & De Smedt, 2017), ordinality judgments (Lyons et al., 2016), or fraction processing (Ischebeck et al., 2009; Jacob & Nieder, 2009). In numerical comparison tasks, for example, activity in the IPS typically increases as the distance between numbers decreases (Ansari, 2008; Hubbard et al., 2005). This "neural distance effect" mirrors the associated "behavioral distance effect" observed in those tasks (i.e., response times increase as the distance numbers decrease; Moyer & Bayer, 1976). It also suggests that quantities may be represented in the IPS along a spatial continuum or "mental

[1] A large body of literature has shown that inferences are influenced by the content of the premises when these are based on real-world content (for a review see Evans, 2003). For example, children and adults find it relatively difficult to infer a conclusion such as "Houses are bigger than skyscrapers" from the premises "Houses are bigger than trailers" and "Trailers are bigger than skyscrapers" because it is inconsistent with prior beliefs. They have to inhibit a belief bias (Houdé, 2019; Moutier et al., 2006). In contrast, it is relatively easy to infer a conclusion such as "Elephants are bigger than mice" from the premises "Elephants are bigger than dogs" and "Dogs are bigger than mice" because it is consistent with prior beliefs. In such cases, however, it is unknown whether participants attend to the logical structure of the arguments or whether their response is based on beliefs about the world.

number line" (Ansari, 2008; Hubbard et al., 2005): the smaller the distance between two quantities on the mental number line, the less distinguishable those quantities are and the longer it may take to compare them.

On the other hand, brain imaging studies indicate that transitive reasoning (typically measured by three-term arguments such as those exemplified in (1) and (2)) also relies on the IPS in adults (Prado et al., 2011). That is, over and above differences in experimental materials and procedures between studies, the neuroimaging literature consistently points to enhanced activity in the IPS during transitive reasoning tasks. Interestingly, when multiple premises are used in transitive reasoning tasks (e.g., A>B, B>C, C>D, D>E), a neural distance effect is found in the IPS (Hinton et al., 2010; Prado et al., 2010). That is, activity associated with evaluating pairs increases as the number of intervening items in a pair decreases (e.g., evaluating whether B>C is associated with more activity than evaluating whether B>D). As in numerical comparison tasks, this neural distance effect mirrors a behavioral distance effect that also characterizes transitive reasoning tasks with multiple premises (i.e., response time in evaluating pairs increases as the number of intervening items in a pair decreases) (Potts, 1972, 1974; Prado et al., 2008).

The presence of similar behavioral and neural signatures in number comparison and transitive reasoning tasks (i.e., the behavioral and neural distance effects) might indicate a common underlying mechanism. That is, both numbers and transitive orderings may be encoded along mental representations that may be spatial in nature and supported by mechanisms in the IPS. This hypothesis is supported by a study demonstrating that the exact same region of the IPS exhibits a neural distance effect in numerical comparison and transitive reasoning tasks in adults (Prado et al., 2010). Therefore, the relationship between numerical and transitive reasoning skills may stem from a common reliance on IPS mechanisms supporting the ordering of items along a spatial dimension.

The idea that numerical processing and transitive reasoning would rely on a common mechanism in the IPS predicts that children with impairments in the IPS may exhibit impaired performance on both number processing and transitive reasoning tasks. Two studies confirm this prediction. First, Morsanyi et al. (2013) asked children with dyscalculia to solve linear transitive reasoning problems similar to that in (1). Dyscalculia is a disability affecting the acquisition of numerical and arithmetic skills that has been consistently linked to anatomical and functional impairments in the IPS (Ansari, 2008). The authors found that children with dyscalculia exhibited poor performance (as compared to typically developing children) in transitive reasoning problems with concrete content. In a recent study, Schwartz et al. (2018) further showed that children with dyscalculia struggle to integrate transitive relations such as in (1) and (2), even when the content is abstract and not affected by beliefs. In that study, brain activity was also measured while children were presented with transitive relations. The only region in which less activity was found in children with dyscalculia than in typically-developing children during transitive reasoning was the IPS. Thus, this study provides evidence that the poor transitive reasoning skills of children with dyscalculia may stem from functional impairments in the IPS (see also Schwartz et al., 2020).

How and when does transitive reasoning emerge in children? The first investigations into the development of transitive reasoning skills dates back to Piaget (Piaget, 1952; Piaget & Inhelder, 1967; Wright, 2001, 2012). The tasks used by Piaget to test transitive reasoning involved colored items that were shown to children. Specifically, children were typically presented with two overlapping pairs (e.g., A>B, B>C). They then had to establish the relationship between the two items that were never presented next to one another (e.g., A and C). Using this paradigm, Piaget found that transitive reasoning does not fully emerge until the age of seven or eight years. However, researchers have highlighted several issues with Piaget's methodology (Wright, 2001, 2012). First, when a transitive problem only involves three terms, one of the items in the conclusion is always at the top of the transitive hierarchy (i.e., A), whereas the other is always at the bottom (i.e., C). In other words, children can attach verbal labels to these items (e.g., "A is always the best" and "C is always the worst") and simply use these labels when presented with A and C, without having to engage in genuine reasoning. One simple way to address this caveat is to present participants with at least five premises (A>B, B>C, C>D, and D>E), such that a conclusion that does not involve endpoint items can be tested (i.e., B>D?) (Wright, 2012). Second, it has been argued that Piaget did not adequately ensure that children could remember the premises before evaluating the conclusion (Bryant & Trabasso, 1971). Using non-verbal problems with five premises and an extensive training protocol to ensure that premises were retained, Bryant and Trabasso (1971) demonstrated that children as young as four years could succeed in transitive reasoning tasks (see also Russell et al., 1996). Since that landmark study, transitive reasoning skills have been shown in many animal species, including non-human primates, rats, birds, and fish (Brannon & Terrace, 1998; Grosenick et al., 2007; Paz et al., 2004; Vasconcelos, 2008; see Chapter 7). Therefore, transitive reasoning appears to have a relatively ancient evolutionary history, perhaps because it has critical adaptive value in facilitating the representation of hierarchies in socially organized species (Vasconcelos, 2008).

In sum, there is considerable evidence that, contrary to Piaget's assumptions, transitive reasoning emerges early in children. This does not mean, however, that transitive reasoning skills do not develop throughout elementary school. In fact, the transitive reasoning abilities observed in animals and young children (using paradigms that involve extensive training with non-verbal premises) may be supported by associative learning mechanisms that have little to do with the type of spatial representations thought to underlie transitive reasoning in adults (Frank, 2005; Frank et al., 2003; Vasconcelos, 2008). Indeed, even in paradigms that involve five transitive items (e.g., A>B, B>C, C>D, D>E), the endpoints (A and E) have asymmetric values in the sense that A is always the "best" item and E always the "worst." It is possible that, with extensive training and multiple repetitions of the premises, these asymmetric values transfer to the adjacent items (Delius & Siemann, 1998; Frank et al., 2003; von Fersen et al., 1991). In other words, B might develop a greater associative value than D because B is associated with the "best" item (i.e., A) and D is associated with the "worst" item (i.e., E). The past reinforcement history for each item might then underlie the transitive inference when B is chosen over D in animals and young children (von Fersen et al., 1991). Even though there is convincing evidence that transitive reasoning in adults relies on a representation of items along a spatial continuum, this strategy may not be readily available to young children (who might instead rely on associative learning mechanisms facilitated by a repeated exposition to premises). Thus, the development of transitive reasoning might be characterized by a transition from the use of associative learning mechanisms to a reliance on spatial ordering mechanisms in the IPS. Overall, both numerical processing and transitive reasoning may be characterized by an increase in specialization of the IPS for the representation of ordered information (Prado et al., 2010).

27.2 Analogical Reasoning

Broadly defined, analogical reasoning is the ability to reason with relational patterns (English, 2004). More specifically, analogical reasoning involves abstracting relational patterns and applying them to new entities. A conventional analogy typically takes the form “A is to B as C is to D” (this is formalized “A:B::C:D”) and involves a mapping between some source items (i.e., A and B) and some target items (i.e., C and D). Consider for example the analogy in (3):

(3) Automobile is to gas as sailboat is to?

Solving this analogy first requires reasoners to extract a relational pattern between two items, “automobile” and “gas” (i.e., A and B), before applying this pattern to a third item, “boat” (i.e., C). A probable conclusion can then be generated, i.e., “wind” (i.e., D). In (3), the relation between A and B (“powered by”) is causal but relatively abstract. This makes the analogy more difficult than if it was based on a relation of physical similarity. Consider for example how the solution “melted snowman” comes naturally from the analogy in (4):

(4) Chocolate bar is to melted chocolate as snowman is to?

Therefore, much like transitive reasoning, analogical reasoning requires relational processing. Unlike transitive reasoning, however, the solution of an analogy does not follow out of necessity from the available information. It can only be supported with varying degrees of strength (Bartha, 2013). In that sense, analogical reasoning belongs to the category of inductive reasoning.

Analogical reasoning plays a fundamental role in creativity (Holyoak & Thagard, 1995). As such, analogies have supported a number of important mathematical discoveries over the course of history (Polya, 1954). Consider for example how the famous mathematician Jean Bernoulli used an analogy with the path of the light to solve a classical problem in calculus of variation (the brachistochrone problem) (Polya, 1954; Sriraman, 2005). But analogical reasoning is not only relevant to expert mathematicians when solving complex problems. It is also a critical skill which young children and adolescents may rely on when learning mathematics and solving problems (for a neo-Piagetian theory on analogy through mapping structures in higher cognition, see Halford, 1992; Halford et al., 2010). For example, suppose that some students have used calculus to demonstrate that of all the rectangles with a given perimeter, the one with the greatest area is a square. These students may then infer that of all the boxes with a given surface area, the one with the greatest volume is a cube (Bartha, 2013). This is an example of an analogy in which students recognize a mapping (i.e., a similarity in relational structure) between a source problem (with rectangles and squares) and a target problem (with boxes and cubes). Younger children also rely on analogical reasoning when they are faced with pictorial representations (e.g., pizza slices, number lines) and manipulative materials (e.g., counters, blocks, rods). These have been termed “mathematical analogs” because they also essentially require children to recognize a structural relation between a source (i.e., the pictorial representation of manipulative) and a target (i.e., the mathematical concept to be acquired) (English, 2004). Finally, a relatively underappreciated fact is that teachers commonly use analogies in the classroom to illustrate concepts and procedures (Richland et al., 2004; Richland & Simms, 2015). In other words, analogies are at the heart of mathematics teachers' practices. There are many examples of such analogies. For instance, teachers may use the analogy of balancing a scale to illustrate how two sides of an equation should be equal. They may also use real world situations involving the manipulation of coins or candies to illustrate additive and subtractive concepts (Richland et al., 2004). Overall, there is no doubt that analogical reasoning plays a central role in mathematical learning, both from a learner's and a teacher's perspective.

Several cross-sectional and longitudinal studies indicate that analogical reasoning is related to mathematical development (Fuchs et al., 2005; Green et al., 2017; Primi et al., 2010; Taub et al., 2008). For instance, Fuchs et al. (2005) found that measures of geometric proportional analogies (i.e., sometimes called "matrix reasoning") at the beginning of first grade were related to math outcomes later during the year. Green et al. (2017) further found that a compound measure of relational reasoning (including matrix reasoning) predicted mathematical skills eighteen months later in children and adolescents. Primi et al. (2010) also demonstrated that verbal and spatial analogical reasoning skills were related to growth of mathematical skills from seventh to eighth grade.

Some important support for a foundational role of analogical reasoning in mathematical learning comes from the literature on "patterning," that is, the ability to extract a relational pattern within a given sequence in order to apply this pattern to another sequence (which could have different surface features) (Burgoyne et al., 2017; Rittle-Johnson et al., 2018). In a standard patterning task, children might be presented with alternating shapes of the same color, such as star–circle–star–circle. Children may then be given a set of red and blue squares and be asked to generate a similar pattern. If the pattern from the sequence of shapes is correctly extracted (i.e., A–B–A–B), children can infer the correct sequence of squares of different colors (i.e., red–blue–red–blue). Therefore, a patterning task requires children to recognize the similarity in relational structure (or mapping) between an initial (or source) sequence and a final (or target) sequence. As such, analogical reasoning skills are fundamental to patterning tasks. Cross-sectional and longitudinal studies have found a relationship between patterning performance and mathematical skills (Lee et al., 2011; Pasnak et al., 2016; Rittle-Johnson et al., 2016; Vanderheyden et al., 2011). For instance, Rittle-Johnson et al. (2016) demonstrated that patterning knowledge when children are in first grade predicts their mathematics achievement in fifth grade, independently of a number of cognitive abilities and mathematical skills.

Overall, there is considerable evidence supporting the role of analogical reasoning in mathematical development. As mentioned in the previous paragraphs of this section, however, analogies are not only used by learners but are also frequently employed by teachers to explain concepts and procedures. Interestingly, there is evidence that the quality of analogy-based instructions varies between teachers and that this has an influence of mathematical learning. This is suggested by a cross-cultural comparison of practices in the mathematics classroom (Richland et al., 2007). The authors analyzed videotapes of mathematics teachers in the United States as well as in two Asian regions in which students significantly outperform American students in international measures of mathematical attainment: Hong Kong and Japan. They did not find differences in terms of frequency of use of analogies by teachers across the three regions. However, there were differences in the extent to which these analogies adhered to principles that are known to facilitate and enhance the effectiveness of analogies. For example, teachers in Hong Kong and in Japan made greater use of strategies that enhanced the source of the analogy as compared to teachers in the United States, thereby reducing working-memory demands for students. These included the use of a familiar source analog and the use of visual aids. Analogies from teachers in Hong Kong and in Japan were also more likely to adhere to principles that draw attention to the relational comparison, such as using spatial cues highlighting the mapping between the source and target and using gestures and visualizations. Because students in Hong Kong and Japan typically achieve higher mathematical skills than children in the United States, the study suggests that an efficient use of analogies in the classroom may contribute to mathematical attainment (Richland et al., 2007).

What may be the neurocognitive foundation of the relationship between analogical reasoning and mathematical development? Of course, as is the case for other forms of reasoning, analogical reasoning may relate to mathematical skills because both place great demands on common general cognitive resources, such as working memory (Waltz et al., 2000). However, analogical reasoning is also (like transitive reasoning) a form of relational reasoning. Thus, it may support mathematical development because the ability to understand and manipulate relations is fundamental to mathematical knowledge. For instance, much like transitive reasoning, analogical reasoning has been found to activate regions in and around the IPS in tasks involving propositional analogies (Bunge et al., 2005; Wendelken et al., 2008) and matrix reasoning (Bunge et al., 2009; Crone et al., 2009; Dumontheil et al., 2010). A recent meta-analysis further found that activity associated with analogical reasoning (and relational reasoning more generally) in the posterior parietal lobe exhibit greater overlap with brain activity associated with numerical processing than with brain activity associated with working memory, attention, or linguistic processing (Wendelken, 2015). This is more consistent with the idea that parietal activity during analogical reasoning may reflect relational computations (a process that is involved in many mathematical tasks) than more general working memory or attentional demands.

The IPS, however, is not the only region involved in analogical reasoning. A large number of studies have also implicated a region located at the apex of the frontal cortex in analogical reasoning, that is, the rostrolateral prefrontal cortex (RLPFC) (Bunge et al., 2005; Wendelken et al., 2008; Wright et al., 2007). The RLPFC and the IPS may have different functional roles in analogical reasoning. For example, some authors have proposed that the IPS is specifically involved in the representation of relations (Singley & Bunge, 2014). In contrast, the RLPFC should support the integration of different mental relations (Bunge & Wendelken, 2009; Wendelken et al., 2008). Evidence for this idea comes from studies showing that the RLPFC is more active when participants compare the relations between two pairs of words (i.e., a classic analogy task) than when they process relations independently from one another (i.e., when they do not have to compare the relations) (Bunge et al., 2005; Wendelken et al., 2008). Interestingly, activity in the RLPFC does not vary with the associative strength of the relationship between words, suggesting that it is not involved in the retrieval of relations per se, but rather in their integration (Bunge et al., 2005; Wendelken et al., 2008). These findings on conventional analogies are consistent with studies on matrix reasoning, which also point to increased activity in the RLPFC when multiple geometric relations have to be considered jointly (as compared to the processing of one single relation) (Dumontheil et al., 2010). Therefore, neuroimaging studies suggest that a fronto-parietal network that includes the IPS (as well as neighboring parietal regions) and the RLPFC supports analogical reasoning.

It is interesting to note that the RLPFC develops more slowly than most other brain regions, only reaching maturity after adolescence (Dumontheil et al., 2008). In line with this observation, developmental studies have found increases of activity in the RLPFC (as well as in the IPS) from childhood to adolescence in analogical reasoning tasks (Crone et al., 2009; Dumontheil et al., 2010; Wright et al., 2007). More specifically, the RLPFC should become increasingly specialized for higher-order relational processing with age. For example, Wright et al. (2007) presented children from six to thirteen years and adults with visual analogy trials in which participants had to indicate which of four objects would complete the problem ("chalk is to chalkboard as pencil is to?). These trials (in which two relations have to be compared) were compared to semantic trials in which participants had to choose among several objects the one that was the most closely semantically related to a cued object. The results indicate age-related increases of activity in the RLPFC for both analogy and semantic trials in children. In contrast, adults showed increased activity in the RLPFC (as a function of accuracy) in analogy but not semantic trials. Using another visual analogy task, Wendelken et al., 2011) further found that activity in the RLPFC distinguishes between trials that require a comparison between two relations (i.e., an analogical judgment) and trials that only require the processing of a single relation in adolescents. However, no such difference was found before the age of fourteen years. Interestingly, that study showed that the RLPFC and the IPS are interconnected during the development of analogical reasoning. That is, the degree of specialization of the RLPFC for relational integration was associated with the degree of cortical thinning in the parietal cortex. Therefore, as pointed out by Singley and Bunge (2014), structural development in the parietal lobe promotes RLPFC selectivity for higher-order problems, perhaps in that a more mature parietal cortex can complete lower-order tasks without taxing frontal regions.

Overall, the relatively delayed development of the brain system supporting analogical reasoning may explain some of the development of analogical reasoning skills in children. As for transitive reasoning, early findings on analogical reasoning skills of children come from Piaget. Piaget presented children with conventional analogical problems (such as the one in (3)) in a pictorial form. His main finding was that young children typically struggle to solve these problems, often relying on physical similarities between items rather than on relations between pairs of items (Inhelder & Piaget, 1958). This led Piaget to suggest that children might not be able to solve these types of problems before the age of eleven or so (Inhelder & Piaget, 1958). More recently, studies have found that the ability to solve classic analogical problems emerge much earlier, to the extent that children are familiar with the relations involved (Goswami & Brown, 1989; Richland et al., 2006; Singer-Freeman, & Goswami, 2001; Tunteler & Resing, 2002). For example, children as young as three years may be able to solve an analogy based on relations of physical causality, such as the one in (4) (Goswami & Brown, 1989). However, the fact remains that young children's analogical reasoning abilities are limited and only slowly improve through childhood and adolescence (such that only adolescents and adults may be able to solve the more abstract analogy in (3)). How can one explain this development? Clearly, one needs to have knowledge about the world to abstract relations and reason analogically. Therefore, with increasing knowledge, the relational similarity between different pairs of items should become increasingly salient (Rattermann & Gentner, 1998; Vendetti et al., 2015). Of course, because knowledge about the world increases with age, this may explain the increase in analogical reasoning performance with age.[2] Another ability that is likely to be important to improved analogical reasoning performance is the capacity to ignore information that is not relevant to the task, such as the physical similarities between items. Studies have shown that young children are very susceptible to distracting information in analogical reasoning tasks (Richland et al., 2006). Inhibiting such information is likely to require efficient executive control skills (including working memory and inhibition, Houdé, 2000, 2019), which also increase with age. Overall, the fact that increases in analogical reasoning performance with age is characterized by (i) a decrease in focus of similarities between items and (ii) an increase in focus on relational information between pairs of items is often characterized as a "relational shift" (Rattermann & Gentner, 1998). Before this relational shift, analogical reasoning remains difficult for young children. This is especially the case when relational patterns are more conceptual than perceptual, which is likely to be the case in many aspects of mathematical learning. Therefore, as stated by Vendetti et al. (2015, p. 102), "research suggests that elementary school children may need structured guidance when attempting to make relational comparisons between domains so that they draw the intended conclusion from the analogy."

27.3 Implications for the Classroom: The Role of Problem-Solving

As is made clear by the literature review in Section 27.2, there is little doubt that relational reasoning skills contribute to mathematical growth in children. Yet, we also reviewed evidence indicating that some of those skills only slowly develop from early childhood to late adolescence. Thus, education may play an important role in scaffolding children's relational reasoning abilities throughout the mathematics curriculum. An interesting way for teachers to promote reasoning in the classroom may be to have children engage in problem-solving, that is, "the cognitive process directed at achieving a goal when the problem solver does not initially know a solution method" (Mayer & Wittrock, 2006, p. 287). Indeed, following pioneering work by Lakatos (1976), Polya (1945), and Schoenfeld (1985), research in math education has often suggested that problem-solving may be an effective way of promoting mathematical reasoning (Törner et al., 2007). We point to five reasons for this. First, problem-solving allows students to apply mathematical knowledge and, in doing so, makes learning meaningful (Artigue & Houdement, 2007). For example, Gibel (2013) argues that solving problems is a situation in which students have to engage in reasoning processes, but also have to assess the validity of these processes. Second, problem-solving provides a context in which reasoning may serve different goals. For example, reasoning may support decision-making or the development of a general solving method (starting, for example, from specific instances). It may also promote arguments regarding the validity and relevance of the results. Third, problem-solving is an opportunity for students to encounter different modes of reasoning (Dias & Durand-Guerrier, 2005; Douek, 2010; Grenier, 2013). For

[2] Note, however, that this idea assumes that increases in performance stem from knowledge acquisition per se rather than from chronological age.

example, Gardes and Durand-Guerrier (2016) have shown that experimental approaches to mathematics learning may involve both deductive and inductive reasoning. An experimental approach to mathematics learning may also highlight what is a cornerstone of mathematical thinking, that is, the interplay between mathematical knowledge and heuristic processes (Polya, 1954). Fourth, problem-solving may encourage proof thinking in students (Balacheff, 1988). Finally, solving problems in the classroom is an ideal situation for students to engage in scientific debates, which may be beneficial to reasoning skills and critical thinking in general (Brousseau, 1997; Douek, 2010; Kuhn & Crowell, 2011).

Problems in the classroom may have different learning goals, and therefore may engage reasoning in different ways. It may be useful to break down mathematical problems into three different categories, depending on their learning goals: problems that focus on acquiring new concepts, problems that focus on strengthening already acquired concepts, and problems that focus on promoting investigative processes *in themselves*. Whereas the explicit goal of the first two types of problems may be to learn and practice some specific reasoning skills, the last type of problems may incidentally engage reasoning skills while students work out the solution of the problem. Consider for example the problems in (5) and (6):

(5) A teacher gives children six strips of paper, each of a different length. The children have to order these strips from the smallest to the largest. After they are given the opportunity to work on the problem, the teacher points to the fact that one can start by putting down the smallest of the six strips, then (next to it) the smallest of the remaining five strips, and so on.

(6) A teacher gives children several nesting cups (not nested) and a suitcase. Children have to store the cups in the suitcase. Because the suitcase is relatively small, the only way to store the cups is to nest them from the smallest to the largest.

The situation in (5) is an example of a problem whose explicit goal is to learn a new concept (i.e., serial order), using a particular technique. In contrast, the goal of the situation in (6) is merely to investigate and look for a solution. However, in doing so, children may apprehend serial order as a relevant reasoning strategy to solve the problem. Note that, in (5) as in (6), students can manipulate the materials to develop some reasoning and come up with a solution. They implement what can be described as an "experimental approach," in other words, a "back-and-forth between manipulation of objects and theoretical elaborations realized through the articulation of three processes: experimentation, formulation, and validation" (Gardes, 2018, p. 83). Such an experimental approach consists of making conjectures, testing them, modifying them and proving those that have withstood the test. This not only promotes inductive reasoning when student have to make conjectures, but also deductive reasoning when those conjectures have to be formally demonstrated (Polya, 1954).

Although having students to engage in problem-solving situations may be a prerequisite to promote reasoning skills in the classroom, it may not be sufficient. That is, for learning to occur, situations may need to adhere to some criteria. For example, students should work on problems by themselves, without the teacher's intervention (Brousseau, 1997). Problems should also (i) be challenging, (ii) concern a conceptual domain that is familiar to the students, and

(iii) induce neither the solution nor the solving method. Therefore, teachers should create an environment that encourages students to become involved in problem-solving.[3] It may also be beneficial for students to work in small groups, so that they can express ideas and explain their reasoning to others. Group work may also encourage students to take into account suggestions from others and encourage argumentation (Mercier et al., 2017). Finally, students may take part in debates. This would expose them to alternative perspectives and force them to engage in a process of formulation and validation of their solutions. Overall, a growing body of literature suggests that group discussions can be a very efficient way to increase reasoning performance in children and adults (Mercier & Sperber, 2011).

There area variety of scenarios in which students may engage in problem-solving, while taking into consideration all of the factors mentioned. However, just to give an example, suppose that (in a first session) students are split into small groups and given some time to solve a mathematical problem (without any intervention from the teacher). The teacher may then ask them to prepare a poster. A follow-up session may then be devoted to the presentation of these posters, as well as to collective discussions through a scientific debate (Arsac & Mante, 2007). Although this situation is just one among many possible scenarios, regular use of such problem-solving sessions in the classroom may help foster reasoning skills.

In Sections 27.3.1–27.3.3, we give some examples of problems that may teach transitive and analogical reasoning in young children. We then give an example of a problem in which both deductive and inductive reasoning are incidentally used when working out the solution.

27.3.1 Learning to Reason with Transitive Relations

Suppose that children are given the following instructions: "James is a tamer of big cats for a circus. James would like his cats to walk in line one behind the other. Can you help James keep his animals in line using the information provided?" (see Figure 27.1).

In order to successfully solve this problem, children must understand the transitive relations in the information provided. For example, from the first two sentences, children can infer the order lion, tiger, and cheetah. The cheetah can then be placed in the second to last position and a possible solution (or all possible solutions) can be proposed. It is easy to change the problem by changing some didactic variables (Brousseau, 1997): the number of animals, the number of constraints, the order of information, the number of correct solutions or the question asked. For example, explicitly asking who follows the lion would force children to use transitive reasoning. Overall, problems such as these may help show children how serial orders can be constructed from transitive relations. They can be solved by children individually or in groups, and even involve materials such as toy animals.

27.3.2 Learning to Reason by Analogy

In a typical patterning problem, children have to describe and reproduce a given alternating sequence (e.g., AABBAABB) using different sets of materials. For instance, children may be presented with a sequence made of alternating white and black diamonds. They are then

[3] By environment, we mean everything that may promote effective learning: material or non-material objects, state of current knowledge, documents, organization of interactions, etc. (Brousseau, 1997).

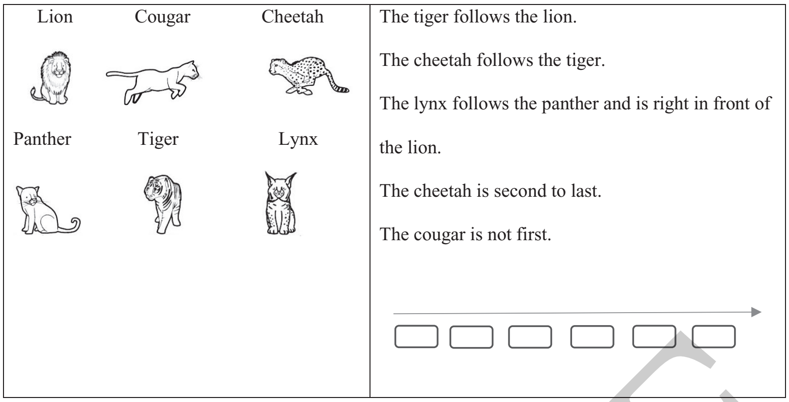

Figure 27.1 Example of materials used in the problem about transitive relations A black-and-white version of this figure will appear in some formats. For the color version, please refer to the plate section.

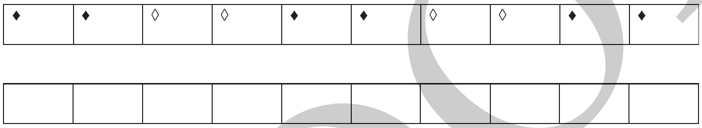

Figure 27.2 Example of materials used in the patterning problem

presented with an empty grid and given the following instructions: “Look carefully at the design I gave you. When you are done looking at it, hide it. Then, try to reproduce that design on the grid, with your tokens. When you think you are done, compare your design with the one I gave you” (see Figure 27.2). Children may work on such problems alone.

In order to successfully solve such a patterning problem, students need to notice what is unique about the design (i.e., the alternating sequence), so they can replicate it. Children who recognize the alternating pattern of two black diamonds and two white diamonds will succeed in reproducing the sequence with the same set of materials (black and white diamonds) but may also abstract the sequence if given a different set of materials (e.g., blue and yellow squares). In other words, several didactic variables can be manipulated, such as the pattern, the number of different tokens (shape and/or color), the initial design viewing time, the presence of a grid, etc. The teacher may also encourage children to explicitly formulate the pattern by asking them to tell other children how to make the design.

The two problem-solving situations described here explicitly aim to teach relational reasoning. But even problems that may not necessarily be used to explicitly teach reasoning (e.g., problems that are used to promote investigative behavior) may encourage students to reason. For instance, these problems may highlight the difference between inductive reasoning (e.g., analogical reasoning) and deductive reasoning (e.g., transitive reasoning). An example of one such problem is given here.

27.3.3 Learning to Investigate: The Interplay between Inductive and Deductive Reasoning

Consider the following scenario (Aldon et al., 2017):

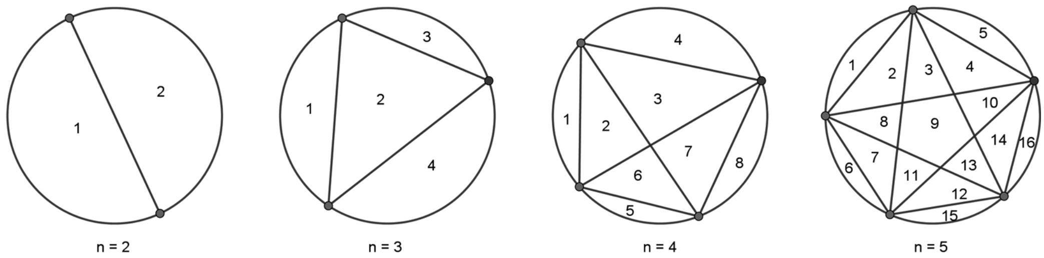

Figure 27.3 Examples of line configurations with different values of n in the geometry problem. For n = 2 there are two areas, for n = 3 there are four areas, for n = 4 there are eight areas, and for n = 5 there are sixteen areas. From these observations, students may use inductive reasoning to make the following conjecture: For n points on the circle, the number of areas is 2^{n-1}. However, perseverant students will realize that drawing lines with n = 6 leads to thirty-one areas (instead of thirty-two areas)! Thus, the formula must be incorrect

A black-and-white version of this figure will appear in some formats. For the color version, please refer to the plate section.

(7) When two points are on a circle, the line joining them defines two areas within the disk. How about with three points? four points? and n points? What is the maximum number of areas possible within the disk?

A first step toward solving this problem often consists of drawing lines with the first few values of n. Figure 27.3 displays the number of areas corresponding to values of n from two to five.

One method that can be used to explain this surprising result is to come up with a formula after having systematically studied and enumerated several geometric configurations. For example, at the very beginning, there is only one point on the circle and only one area (i.e., the whole disk). Drawing another point (n = 2) adds a line, which itself adds an area (see Figure 27.3). Therefore, when n = 2, there is one line and two areas. With n = 3, there are three lines and each line adds one area. Thus, there are 1 + 3 = 4 areas. Students may then start to hypothesize that counting the areas is equivalent to counting the lines. In other words, it amounts to counting the combinations of two out of n points, i.e., $\binom{n}{2} = \frac{n(n-1)}{2}$. With n = 4, however it becomes clear that this reasoning is insufficient. Indeed, there is one additional region every time two lines intersect. In fact, there are as many points of intersection as there are quadrilaterals whose corners are among the n points on the circle, that is, the number of combinations of four points among n: $\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4!}$. From this observation, students can derive the general formula of the maximum number of regions determined by n points on the circle: $1 + \binom{n}{2} + \binom{n}{4} = 1 + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{4!}$.

Overall, this situation illustrates the necessary interplay between inductive and deductive reasoning. Inductive reasoning is useful to make progress when looking for the solution of a problem. For example, it can be used to make conjectures. However, inductive reasoning does not always make it possible to validate the solution. Here, for instance, studying the geometric configuration with n = 6 makes it clear that the conjecture is wrong. But it does not explain why. Students need to use deductive reasoning when systematically studying how areas are added when points are drawn on the circle. This makes it possible to (i) find the correct solution given by a formula and (ii) demonstrate this formula.

27.4 Conclusion

Over the past decades, the literature on mathematical cognition has largely focused on the idea that primitive intuitions about quantities serve as a foundation for symbolic mathematics. This has notably led to the development of theories emphasizing how individual differences in the quality of magnitude representations affect math learning (Feigenson et al., 2013). Although this focus has allowed for significant progress to be made in the field, it is also clear that mathematical skills are wide ranging and go beyond basic representations of numerical magnitudes. In other words, mathematical skills are likely to involve multiple cognitive processes and representations. In the present chapter, we argue that the ability to understand and integrate relations is central to the development of mathematical skills. We have reviewed research indicating that such relational reasoning skills may be present early in development, but also considerably improve from childhood to adolescence. Thus, mathematics teachers may have an important role to play in nurturing relational reasoning skills in children. To reach this educational goal, they may use problem-solving situations in which students are confronted to different forms of relational reasoning as well as to the fundamental difference and complementarity between induction and deduction.

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